RESEARCH STATEMENT

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1. Introduction

My Ph.D. thesis is concerned with the initial value problem for the Benjamin-Ono equation (BO),

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, & x, t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

with $\mathcal{H}$ denoting the Hilbert transform

$$\begin{aligned}\mathcal{H}f(x) &= \frac{1}{\pi}\,\text{p.v.}(\frac{1}{x} * f)(x) = \frac{1}{\pi}\lim_{\epsilon \downarrow 0} \int_{\epsilon<|y|<\frac{1}{\epsilon}} \frac{f(x-y)}{y}dy \\ &= -i\,(\text{sgn}(\xi)\,\widehat{f}(\xi))^{\check{}}(x).\end{aligned}$$

This equation models long internal (gravity) waves traveling through a deep stratified fluid and was deduced by Benjamin (1967) and Ono (1975). It has been shown to be a completely integrable system. The main objective is to study the well-posedness of the IVP for the Benjamin-Ono in the weighted Sobolev spaces $Z_{s,r}$ and $\dot{Z}_{s,r}$ defined as

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \quad s, r \in \mathbb{R}$$

and

$$\dot{Z}_{s,r} = \left\{ f \in Z_{s,r} \ : \ \widehat{f} = 0 \right\},$$

and unique continuation properties of solutions in these spaces while considering the strongest possible decay conditions of solutions to equation (1.1).

Theorem A, stated below, followed the seminal works of R. Iorio, and motivates the results presented here.

Theorem A. *(Fonseca-Ponce [2])*

(i) *Suppose $u \in C([0,T] : Z_{2,2})$ is a solution of the IVP (1.1) such that at two different times, $t_1, t_2 \in [0,T]$*

$$u(\cdot, t_j) \in Z_{5/2,5/2} \qquad j = 1, 2,$$

then $\hat{u_0}(0) = 0$.

(ii) *Suppose $u \in C([0,T] : \dot{Z}_{3,3})$ is a solution of the IVP (1.1) such that at three different times, $t_1, t_2, t_3 \in [0,T]$*

$$u(\cdot, t_j) \in \dot{Z}_{7/2,7/2} \qquad j = 1, 2, 3,$$

then $u(x,t) \equiv 0$.

It is known that we cannot have uniqueness at two different times with $r = 7/2$, that is, the condition involving three times is necessary as it is shown in [3].

The following question presents itself: Can one have uniqueness at two different times by strengthening the decay assumption? The answer is "no," at least not by increasing the decay up to $r = 4$. Furthermore, the following issues arise:

Question 1. *Can $r = 4$ be improved?*

Question 2. *Does more decay ($r > 4$) imply uniqueness under a condition at two times in $\dot{Z}_{s,r}$?*

2. Results

The main result of my dissertation answers Question 1, that is, $r = 4$ *can* be improved, and, in regard to Question 2, it is not the case for $r \in (4, 5]$. In fact, the condition at three times is necessary even if stronger decay is imposed, that is, $r \in [5, 5+1/2^-)$. These results can be found in the article, "On decay properties of solutions to the IVP of the Benjamin-Ono equation,"by Cynthia Flores, published in the Journal of Dynamics and Differential Equations (DOI) 10.1007/s10884-013-9321-6, [1]. The main theorems are now stated:

Theorem 1. *(Flores [1]) There exist infinitely many $u_0 \in \dot{Z}_{7,5}$ such that*

$$\int_{\mathbb{R}} x u_0(x) dx \neq 0,$$

for which the corresponding solution $u \in C(\mathbb{R} : \dot{Z}_{7,7/2^-})$ of the BO satisfies that

$$u(\cdot, t^*) \in \dot{Z}_{7,5},$$

where

$$t^* = -\frac{4}{\|u_0\|_2^2} \int_{\mathbb{R}} x u_0(x) dx.$$

It is important to remember that results of this type with the decay holding only at two different times cannot be true for solutions of other dispersive models, for instance, the generalized KdV and of nonlinear Schrödinger type.

The result in Theorem 1 is shown by considering the time evolution of the moments up to second order of the solutions to (1.1). Furthermore, the condition involving three times is necessary, even if the value of r is increased to $r \in [5, 5 + 1/2^-)$, as Theorem 2, stated below, shows. The proof of Theorem 2 does not rely on the evolution of moments.

Theorem 2. *(Flores [1]) There exist infinitely many $u_0 \in \dot{Z}_{7,5+1/2^-}$ such that*

$$\int_{\mathbb{R}} x u_0(x) dx \neq 0,$$

for which the corresponding solution $u \in C(\mathbb{R} : \dot{Z}_{7,7/2^-})$ of the BO satisfies that

$$u(\cdot, t^*) \in \dot{Z}_{7,5+1/2^-},$$

where t^* is given in Theorem 1.

Remark 1. The techniques used to prove Theorem 1 based on the evolution of the moments up to order two is the best possible since an improvement of the argument shall involve the description of the time evolution of the third moment,

$$\int x^3 u(x,t)dx.$$

However, the most one can show for a nonzero solution is $u \in C(\mathbb{R} : \dot{Z}_{s,7/2^-})$ for any $s > 7/2$ (see [2]).

3. **Overview**

My research area is at the intersection of harmonic analysis, partial differential equations and mathematical physics. Recently, there has been an intense activity in the study of nonlinear dispersive equations. Among the systems considered one finds the Korteweg-de Vries equation, the Schrödinger system and the Benjamin-Ono equation, all arising in different physical problems, mainly nonlinear wave propagation. The Korteweg-de Vries equation describes the evolution of waves traveling through mediums such as shallow water surfaces, plasmas and crystals. Solutions to the Schrödinger system predict the future behavior of a dynamic system, which has important applications in quantum mechanics. As mentioned earlier, the Benjamin-Ono equation models long internal (gravity) waves traveling through a deep stratified fluid. Furthermore, under certain circumstances, these all admit solitary wave solutions called *solitons*, which have important applications in fiber optics, magnetics and genetics.

One particular area of interest regards finding optimal results concerning local and global well-posedness under a minimal regularity requirement of the given data. That is, the long time behavior of local solutions to these dispersive models has been extensively analyzed. Thus the question of the best possible decay space for a general class of solutions to these models presents itself. So it is natural to consider solutions in weighted functions spaces, specifically, $L^2(wdx)$ with weight function w. The aim of my research projects is to **provide optimal unique continuation properties of these dispersive models** that can be deduced by partial information of a solution at two different times.

4. **Future topics of interest**

4.1. **The Benjamin-Ono equation.** It remains unknown whether the condition involving three times described above for the Benjamin-Ono equation can be reduced to two times. It also remains to find whether or not $r = 5 + 1/2^-$ is the best possible for which uniqueness fails at two times. That is, the answers to Question 1 and Question 2 are unknown if we consider values of the parameter $r > 5 + 1/2^-$. Interestingly, a similar unique continuation property is known with a condition involving only two times for the KdV.

4.2. **The dispersion generalized Benjamin-Ono equation.** In addition, questions can be posed for the dispersion generalized Benjamin-Ono equation where the dispersive term describes a continuum of operators between the BO and the KdV equations. Below is the initial value problem for the dispersion generalized Benjamin-Ono equation:

$$\begin{cases} \partial_t u - D^{1+a}\partial_x u + u\partial_x u = 0, \quad x,t \in \mathbb{R} \quad 0 \leq a \leq 1, \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

where D^s denotes the homogeneous derivative of order $s \in \mathbb{R}$,

$$D^s = (-\partial_x^2)^{s/2} \quad \text{and} \quad D^s = (\mathcal{H}\partial_x)^s,$$

and $D^s f = c_s(|\xi|^s \widehat{f})\check{}$.

When $a = 1$, equation (4.1) becomes the Korteweg-de Vries equation and when $a = 0$, equation (4.1) is the Benjamin-Ono equation. By the nature of the Fourier transform, there is an opportunity to discover regularity properties of solutions to the dispersion generalized Benjaimin-Ono equation that arise under decay assumptions. This is an area I am interested in exploring.

4.3. **Harmonic analysis.** Ongoing discussions include the study of further techniques in harmonic analysis (for instance, multi-linear estimates) which can assist in gaining a better understanding of the qualitative behavior of solutions to nonlinear dispersive models. I believe that tools coming from harmonic analysis will continue to provide new key arguments to answer several open questions appearing in nonlinear dispersive equations. For these reasons, I plan to continue studying problems and new techniques in harmonic analysis.

5. **Conclusion**

As I move forward with my research agenda, I expect to widen the understanding in regard to unique continuation of solutions of the Benjamin-Ono equation and to reveal properties of solutions to the dispersion generalized Benjamin-Ono equation and to broaden my research interests in harmonic analysis.

References

[1] C. Flores, (2013) *On decay properties of solutions to the IVP of the Benjamin-Ono equation,* J. Dyn. Diff Eq, (DOI) 10.1007/s10884-013-9321-6

[2] G. Fonseca and G. Ponce, (2011) *The I.V.P for the Benjamin-Ono equation in weighted Sobolev spaces*, J. Func. Anal **260** 436-459.

[3] G. Fonseca, F. Linares, G. Ponce, (2012) *The I.V.P for the Benjamin-Ono equation in weighted Sobolev spaces II,* J. Func. Anal **262** 2031-2049.

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